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ORIGINAL PAPER

One-dimensional trapped atoms: critical coupling parameter and critical number of particles

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Abstract In this paper we consider the case of a Bose gas in low dimension in order to illustrate the applicability of a method that allows us to construct analytical relations, valid for a broad range of coupling parameters, for a function which asymptotic expansions are known. The method is well suitable to investigate the problem of stability of a collection of Bose particles trapped in one- dimensional configuration for the case where the scattering length presents a negative value. The eigenvalues for this interacting quantum one-dimensional many particle system become negative when the interactions overcome the trapping energy and, in this case, the system becomes unstable. Here we calculate the critical coupling parameter and apply for the case of Lithium atoms obtaining the critical number of particles for the limit of stability.

Keywords One-dimensional trapped atoms · Critical coupling parameter

1 Introduction

The topic of low-dimension quantum gases has a long history of scientific and mathematical interest [1,2]. Dimensionality is a characteristic that determines many of the important properties of physical systems. For degenerate quantum gases, the dimensionality plays the dominant rule in determining the existence of a transition known as Bose-condensation. While for a long time the discussion of low-dimension

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system was mostly academic, recently this picture has suffer a dramatic change. The quite fast progress in experimental system turned possible to perform real experiment employing low dimensionality. The capability to tight confining the motion of atoms in one or two of the trapping dimensions, allow creating system very close to two or one dimension. The production of kinematically low dimension quantum gases has revival old interest with new faces now. One of such topics is the rule of interparticle interactions on the behavior of the system. Effects of finite size and finite temperatures still present open questions.

Interactions between particles drastically changes the conventional pictures associated to Bose-Einstein condensation. For a $T = 0$ approximation, the harmonic trapped one-dimensional (1D) gas presents different regimes according to the number of atoms and potential dimensions. A Gaussian profile, a Thomas–Fermi profile or a Tonks gas are all possible.

Generically, there are many effects related to the 1D aspects of confinement. Solitons are a good example of such effects. These types of systems comprise examples of one-dimensional quantum gases. These have the important property that in many cases their energy eigenstates are exactly soluble, resulting in a great increase of fundamental understanding of many-body quantum field theory. For this reason, the study of these systems plays an important role in the physics of quantum many-body systems. It is possible, in principle, to make first-principle predictions without introducing added approximations like perturbation theory. This permits a direct experimental test of the underlying many-body quantum physics.

When considering the properties of trapped atoms at arbitrary coupling parameters, one usually needs to invoke computer calculations. Analytical expressions can be available only in the weak-coupling and strong-coupling limits. Nevertheless, there is a method that permits to reconstruct an analytical formula that is valid for the whole region of coupling parameters for a function whose asymptotic expansions in the weak-coupling and strong-coupling limits are known.

It is possible to quote hundreds of examples of different crossovers. In fact, many physical quantities qualitatively change their behavior when passing from the weak-coupling to the strong-coupling limit [3], for instance the majority of problems having the behavior of energies as functions of a coupling parameter in statistical physics, quantum mechanics and field theory. Let us mention on this respect the dependence of the spectra of Schrödinger operators on the anharmonicity parameter for several anharmonic models. The energy spectrum of such models is qualitatively different in the weak-coupling (weak anharmonicity) as compared to the strong-coupling (strong anharmonicity) limits.

Here we delineate self-similar root approximants and we apply them for describing several properties of one-dimensional trapped atoms.

2 Self-similar root approximants

Consider that we are interested in the behavior of a function $f(s)$ of the coupling parameter s. Let this function be defined by a complicated equation that can be solved only numerically. Nevertheless, we can find the asymptotic expansion

$$f(s) = a_0 + a_1 s + a_2 s^2 + ... \ (s \to 0) \tag{1}$$

in the weak-coupling limit. Often, we can also derive the asymptotic expansion

$$f(s) = b_0 s^{\beta_0} + b_1 s^{\beta_1} + b_2 s^{\beta_2} + ... (s \to \infty) \tag{2}$$

in the strong-coupling limit, where the powers β_j are arranged in the decreasing order, $\beta_j > \beta_{j+1.}$

Introducing in series (1) control functions by means of an algebraic transformation and using the self-similar approximation theory [4–16], we obtain the self-similar root approximant [17,18]

$$f_k^* (s) = a_0 \left(... \left\{ \left[(1 + A_1 s)^{n_1} + A_2 s^2 \right]^{n_2} + A_3 s^3 \right\}^{n_3} + \cdots + A_k s^k \right)^{n_k}, \tag{3}$$

where k is the order of the approximation taken. The coefficients A_j and powers n_j are to be defined by considering the strong-coupling limit of the approximant in Eq. (3) and equating it to the strong-coupling expansion in Eq. (2). This way can be called the left-to-right crossover.

In general, it could be possible to go to the opposite way, i.e., from the right to the left. That is, we could construct a nested-root approximant starting from the strong-coupling asymptotic form of Eq. (2) and then define the corresponding coefficients and powers by equating to the asymptotic expansion of Eq. (1). However, the right-to-left crossover results in approximants usually are less accurate than the left-to-right crossover formulas. This is connected to the fact that the weak-coupling expansions have, as a rule, zero radius of convergence, while the strong-coupling ones have a finite radius of convergence. The accuracy of the left-to-right crossover approximants is usually better than that of the right-to-left ones because of the larger region of applicability of the strong-coupling expansion in Eq. (2) as compared to the region of validity of the weak-coupling expansion in Eq. (1). In fact, the latter can be valid for $s << 1$, hence its region of validity is inside the interval [0,1). In contrast, the strong-coupling form, derived for $s >> 1$, has the region of applicability inside the interval $(1, \infty)$. Therefore, the self-similar crossover approximant has to be fitted to the asymptotic expansion that possesses the larger region of validity.

When considering the strong-coupling limit $s \to \infty$ for the approximant of Eq. (3), we need to know the relation between the powers n_j and the numbers $j = 1, 2...$ Among all possible relations, we have to choose that one which is the most general, imposing no restrictions on the powers β_j. It is possible to show that the condition

$$j n_j < j + 1 \quad (j = 1, 2, ..., k - 1) \tag{4}$$

provides a general way of expanding the form of Eq. (3), valid for any $k = 1, 2, \ldots$ and any arbitrary β_j.

Under the criterion of Eq. (4), and rewriting the approximant in Eq. (3) in the form

$$f_k^*(s) = a_0\left(A_k s^k\right)^{n_k}\left(1+\frac{A_{k-1}^{n_{k-1}}}{A_k}x^{k-(k-1)n_{k-1}}\times\left\{1+\frac{A_{k-2}^{n_{k-2}}}{A_{k-1}}x^{k-1-(k-2)n_{k-2}}\right.\right.$$
$$\left.\left.\times\left(1+\ldots+\frac{A_2^{n_2}}{A_3}x^{3-2n_2}\times\left[1+\frac{A_1^{n_1}}{A_2}x^{2-n_1}\left(1+\frac{x}{A_1}\right)^{n_1}\right]^{n_2}\right)^{n_3}\ldots\right\}^{n_{k-1}}\right)^{n_k}$$

where x $\equiv$ s^{-1}, it is easy to expand the latter in powers of x. Comparing the resulting expansion with the strong-coupling limit in Eq. (2) we obtain

$$\begin{aligned} kn_k &= \beta_0, \\ (k-j)\,n_{k-j} &= \beta_j - \beta_{j-1} + k - j + 1, \end{aligned} \tag{5}$$

with $1 \le j \le k-1$. The values of n_j, defined by Eqs. (5), are in compliance with the criterion showed in Eq. (4) because of the inequality $\beta_j - \beta_{j-1} < 0$.

The first-order self-similar approximant in Eq. (3) is

$$f_1^*(s) = a_0\,(1+As)^{n_1},$$

where

$$A^{n_1} = \frac{b_0}{a_0}, n_1 = \beta_0.$$

The second-order approximant in Eq. (3) takes the form

$$f_2^*(s) = a_0\left[(1+A_1 s)^{n_1} + A_2 s^2\right]^{n_2},$$

in which

$$\begin{gathered} A_1^{n_1 n_2} = \frac{b_0}{a_0}\left(\frac{b_1}{n_2 b_0}\right)^{n_2}, \quad A_2^{n_2} = \frac{b_0}{a_0}, \\ n_1 = \beta_1 - \beta_0 + 2, \quad 2n_2 = \beta_0 \end{gathered}$$

In the third order, we find

$$f_3^*(s) = a_0\left\{\left[(1+B_1 s)^{n_1} + B_2 s^2\right]^{n_2} + B_3 s^3\right\}^{n_3},$$

where

$$B_1^{n_1 n_2 n_3} = \frac{b_0}{a_0}\left(\frac{b_1}{n_3 b_0}\right)^{n_3}\left(\frac{b_2}{n_2 b_1} - \frac{n_3-1}{2n_2 n_3}\frac{b_1}{b_0}\right)^{n_2 n_3},$$
$$B_2^{n_2 n_3} = \frac{b_0}{a_0}\left(\frac{b_1}{n_3 b_0}\right)^{n_3}, B_3^{n_3} = \frac{b_0}{a_0},$$
$$n_2 = \beta_2 - \beta_1 + 2, \quad 2n_2 = \beta_1 - \beta_0 + 3, \quad 3n_3 = \beta_0.$$

The method of constructing self-similar crossover formulas is also applicable to asymptotic expansions more general than showed in Eq. (1), for instance to series

$$f(s) = a_0 + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots \quad (s \to 0), \tag{6}$$

in which α_j are arbitrary positive powers arranged in the increasing order as

$$0 < \alpha_j < \alpha_{j+1.} \tag{7}$$

Then, instead of Eq. (3), we obtain the self-similar approximant

$$f_k^*(s) = a_0\left(\ldots\left\{\left[\left(1 + A_1 s^{\alpha_1}\right)^{n_1} + A_2 s^{\alpha_2}\right]^{n_2} + A_3 s^{\alpha_3}\right\}^{n_3} + \cdots + A_k s^{\alpha_k}\right)^{n_k}. \tag{8}$$

The criterion showed in Eq. (4) transforms to the inequality

$$\alpha_j n_j < \alpha_{j+1}. \tag{9}$$

And, in the place of Eqs. (5), we find

$$\begin{aligned} \alpha_k n_k &= \beta_0, \\ \alpha_j n_j &= \alpha_{j+1} + \beta_{k-j} - \beta_{k-j-1}, \end{aligned} \tag{10}$$

with j = 1, 2, ..., $k - 1$.

The described method makes it possible to construct analytical interpolative formulas for the whole range of the coupling parameter. The method can also be used for interpolating any functions of other variables, provided the corresponding asymptotic expansions are available.

3 One-dimensional confined system

To illustrate the presented method, let us consider a model of a one-dimensional system of harmonic trapped atoms [17]. This means that in the eigenproblem $\mathrm{H}\,\psi = \mathrm{E}\psi$, we consider the nonlinear Hamiltonian

$$\mathrm{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - g\,|\psi|^2, \tag{11}$$

in which $x \in (-\infty, \ +\infty)$.

In order to derive the weak-coupling and strong-coupling asymptotic expansions, we may resort to the optimized perturbation theory [19,20]. To this end, we start with the trial Hamiltonian

$$\mathrm{H}_0 = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{u^2}{2}x^2, \tag{12}$$

containing a control parameter, u, and possessing the eigenvalues

$$\mathrm{E}_n^{(0)} = \left(n + \frac{1}{2}\right)u,$$

and having the eigenfunctions

$$\psi_n^{(0)}(x) = \frac{(u/\pi)^{1/4}}{\sqrt{2^n n!}}\mathrm{H}_n\left(\sqrt{u}x\right)\exp\left(-\frac{1}{2}ux^2\right),$$

where $n = 0, 1, 2, \ldots$

The first-order approximation gives

$$\mathrm{E}_n^{(1)}(g, u) = \frac{u}{2}\left(n + \frac{1}{2}\right)\left(1 + \frac{1}{u^2}\right) - \sqrt{u}I_n g, \tag{13}$$

with the notation

$$I_n \equiv \frac{2}{\pi\,(2^n n!)^2}\int_0^\infty \mathrm{H}_n^4(x)e^{-2x^2}dx.$$

The variational condition for Eq. (13) yields

$$u^2 - su^{3/2} - 1 = 0 \tag{14}$$

for the control function $u = u(s)$, where

$$s \equiv \frac{2I_n g}{2n+1}. \tag{15}$$

For weak coupling, when $s \to 0$ and $g \to 0$, the control function is

$$u \approx \frac{1}{(2n+1)^3} \times \left[8n^3 + 12n^2 + 6n + 1 - 4I_n gn^2 - 4I_n gn - I_n g - 2g^2 n - g^2 + \frac{7}{8}I_n g^3\right]. \tag{16}$$

For strong coupling, when $s \to \infty$ and $g \to \infty$, the control function is

$$u \approx \frac{1}{2}(2)^{1/3}\left(I_n[2n+1]\right)^{2/3}\left(\frac{1}{g}\right)^{2/3}+\frac{2n^2}{3g^2}+\frac{2n}{3g^2}+\frac{1}{6g^2}$$
$$-\frac{\frac{7}{24}I_n 2^{2/3}\left(I_n[2n+1]\right)^{1/3}\left(\frac{1}{g}\right)^{1/3}n}{g^3}-\frac{\frac{7}{144}I_n 2^{2/3}\left(I_n[2n+1]\right)^{1/3}\left(\frac{1}{g}\right)^{1/3}}{g^3}$$
$$-\frac{\frac{7}{18}I_n 2^{2/3}\left(I_n[2n+1]\right)^{1/3}n^3}{g^3}-\frac{\frac{7}{12}I_n 2^{2/3}\left(I_n[2n+1]\right)^{1/3}\left(\frac{1}{g}\right)^{1/3}n^2}{g^3} \quad (17)$$

For the optimized approximant

$$\mathrm{E}(s) \equiv \mathrm{E}_n^{(1)}(g(s), u(s)), \quad (18)$$

we find

$$\mathrm{E}(s) = \frac{1}{2}\left(n+\frac{1}{2}\right)\left(\frac{3}{u}-u\right). \quad (19)$$

Equation (19), together with the control-function showed in Eq. (14), results in the weak-coupling expansion of spectrum when $g \to 0$ in Eq. (18)

$$E(g) \approx \frac{\left(n+\frac{1}{2}\right)\left(\begin{array}{l}\frac{11}{324}2^{1/3}\left(I_n[2n+1]\right)^{2/3}+\frac{44}{81}2^{1/3}\left(I_n[2n+1]\right)^{2/3}n^4+\frac{88}{81}2^{1/3}\left(I_n[2n+1]\right)^{2/3}n^3\\+\frac{22}{27}2^{1/3}\left(I_n[2n+1]\right)^{2/3}n^2+\frac{22}{81}2^{1/3}\left(I_n[2n+1]\right)^{2/3}n\end{array}\right)}{2g^{14/3}}$$
$$+\frac{\left(n+\frac{1}{2}\right)\left(I_n 2^{2/3}\left(I_n[2n+1]\right)^{1/3}\left(\frac{7}{18}n^3+\frac{7}{24}n+\frac{7}{144}+\frac{7}{12}n^2\right)\right)}{2g^{10/3}} \quad (20)$$

In the limit of strong-coupling expansion of spectrum, when $g \to \infty$

$$\mathrm{E}(g) \approx \frac{3\left(n+\frac{1}{2}\right)2^{2/3}}{2\left(I_n[2n+1]\right)^{2/3}\left(\frac{1}{g}\right)^{2/3}}+\frac{1}{2}\left(n+\frac{1}{2}\right)$$
$$\times\left(-\frac{6(2)^{1/3}\left(\frac{2}{3}n^2+\frac{2}{3}n+\frac{1}{6}\right)}{\left(I_n[2n+1]\right)^{4/3}}-\frac{1}{2}2^{1/3}\left(I_n[2n+1]\right)^{2/3}\right)\left(\frac{1}{g}\right)^{2/3} \quad (21)$$

In the limit $g \to -\infty$, we have the asymptotic expansion [21]

$$E(g) \approx -\frac{3\left(n+\frac{1}{2}\right)2^{2/3}(-1)^{1/3}}{2\left(I_n[2n+1]\right)^{2/3}\left(-\frac{1}{g}\right)^{2/3}}$$
$$+\frac{1}{2}\left(n+\frac{1}{2}\right)\left(\begin{array}{l}-\frac{6(2)^{1/3}(-1)^{2/3}\left(\frac{2}{3}n^2+\frac{2}{3}n+\frac{1}{6}\right)}{\left(I_n[2n+1]\right)^{4/3}}\\-\frac{1}{2}2^{1/3}\left(I_n[2n+1]\right)^{2/3}(-1)^{2/3}\end{array}\right)\left(\frac{1}{g}\right)^{2/3}. \quad (22)$$

Following Sect. 2, we find the self-similar crossover approximants. In the first order we have

$$\mathrm{E}_1^*(g) = \left(n + \frac{1}{2}\right)(1 + Ag)^{2/3}, \tag{23}$$

with

$$A = \frac{\frac{3\left(n+\frac{1}{2}\right)2^{2/3}}{2(I_n[2n+1])^{2/3}}}{2^{1/3}(I_n[2n+1])^{2/3}\left(\frac{11}{324} + \frac{44}{81}n^4 + \frac{88}{81}n^3 + \frac{22}{27}n^2 + \frac{22}{81}n\right)}.$$

In the second order we find

$$\mathrm{E}_2^*(g) = \left(n + \frac{1}{2}\right)\left[(1 + A_1 g)^{2/3} + A_2 g^2\right]^{1/3}, \tag{24}$$

where

$$A_1^{2/9} = \frac{\frac{3\left(n+\frac{1}{2}\right)2^{2/3}}{2(I_n[2n+1])^{2/3}}}{2^{1/3}(I_n[2n+1])^{2/3}\left(\frac{11}{324} + \frac{44}{81}n^4 + \frac{88}{81}n^3 + \frac{22}{27}n^2 + \frac{22}{81}n\right)} \times \frac{1}{2}\left(-\frac{6(2)^{1/3}\left(\frac{2}{3}n^2 + \frac{2}{3}n + \frac{1}{6}\right)}{(I_n[2n+1])^{4/3}} - \frac{1}{2}2^{1/3}(2I_n n + I_n)^{2/3}\right),$$

$$A_2^{1/3} = \frac{\frac{3\left(n+\frac{1}{2}\right)2^{2/3}}{2(I_n[2n+1])^{2/3}}}{2^{1/3}(I_n[2n+1])^{2/3}\left(\frac{11}{324} + \frac{44}{81}n^4 + \frac{88}{81}n^3 + \frac{22}{27}n^2 + \frac{22}{81}n\right)}.$$

Equations (23) and (24) interpolate between the weak-coupling expansion, Eq. (20), and the strong-coupling limit, Eq. (21).

4 Results and discussion

Varying the coupling parameter g, we can analyze graphically the condensate energy for the self-similar approximation (see Fig. 1). From Fig. 1 we can see that these approximants demonstrate good convergence.

In the region of $g < 0$, the energy showed in Eq. (13) is positive for small $|g|$, and, as g diminishes, the energy becomes zero at a critical value g_c. The latter can be found from the definition

$$\mathrm{E}(g_c) = 0. \tag{25}$$

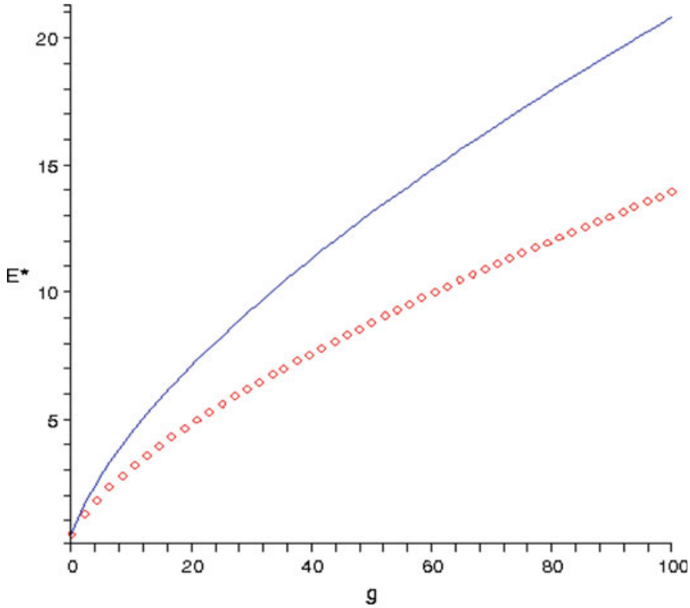

Fig. 1 The values of the coupling parameter g as function of different condensate energies for the self-similar approximation

The form showed in Eq. (19) shows that the equality showed in Eq. (25) holds true for $u_c^2 = 3$. Then, Eq. (14) immediately gives

$$s_c = -\frac{2}{3^{3/4}} = -0.87738. \tag{26}$$

Due to Eq. (15), one has

$$g_c = -(2n+1)\frac{s_c}{2I_n} \quad . \tag{27}$$

For example, for the ground-state level, with $n = 0$, one finds

$$g_c = -1.0996327. \tag{28}$$

For $g < g_c$, the energy becomes negative, which implies the instability of the system.

The fact that there is a critical value for the coupling parameter $g \equiv 4\pi \frac{a_s}{l_0} N$ defines the critical number of atoms [22]

$$N_c = \frac{l_0 g_c}{4\pi a_s} \tag{29}$$

For the trapped atoms of ^{7}Li, having negative scattering length $a = -1.5 \times 10^{-7}$cm, and the oscillator length $l_0 = 3.160 \times 10^{-4}$cm, we get $N_c \approx 200$ atoms.

Others current theories predict that the Bose-Einstein condensation (BEC) can occur in different traps calculated with no more than 1400 condensate atoms [23]. BEC with attractive interactions have been realized with ^{7}Li by Bradley et al. [24] and measurement of the maximum critical number of atoms, N_c, in the condensate, in spherical trap, is in good agreement with values predicted theoretically, within experimental uncertainties. In our case, we calculated the critical number of atoms, N_c, for one-dimensional trapped atoms, because we have obtained the N_c less than others theoretical results determined in spherical trap.

5 Conclusion

The self-similar root approximants permit one to construct accurate analytical expressions for energy levels as well as for wave functions. The method can find numerous practical applications, for example, analyzing spectral properties of atoms and molecules, and investigating the behavior of Bose-condensed gases in magnetic traps. By employing the self-similar approximation theory, we have found the analytical solution for one-dimensional trapped atoms.

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